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A Note on Some Recent Strong Convergence Theorems of Iterative Schemes for Semigroups with Certain Conditions

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Abstract. In this note, suggesting an alternative technique we partially modify and fix the proofs of some recent results focused on the strong convergence theorems of iterative schemes for semigroups including a specific error observed frequently in several papers during the last years. Moreover, it is worth mentioning that there is no new constraint invloved in the modification process presented throughout this note.

Keywords: Nonexpansive semigroups · Strong convergence
Variational inequality · Strict pseudo-contraction
Strictly convex Banach spaces · Fixed point

1 Introduction

Throughout this note, we suppose that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\mathbb{R}^+$ and $\mathbb{N}$ are the set

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of nonnegative real numbers and positive integers, respectively. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \ \forall x \in E$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized pairing. It is well-known that if E is smooth, then J is single-valued, which is denoted by j.

Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T. If $\{x_n\}$ is a sequence in E, we use $x_n \to x$ ($x_n \rightharpoonup x$) to denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

Recall that a mapping $f : C \to C$ is called a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$||f(x) - f(y)|| \leq \alpha ||x - y||, \ \forall x, y \in C.$$

We use $\prod_C$ to denote the collection of mappings f satisfying the above inequality.

$$\prod_C = \{f : C \to C \mid f \text{ is a contraction with some constant } \alpha\}.$$

Note that each $f \in \prod_C$ has a unique fixed point in C, (see [1]). And note that if $\alpha = 1$ we call nonexpansive mapping.

Let H be a real Hilbert space, and assume that A is a strongly positive bounded linear operator (see [2]) on H, that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \overline{\gamma} \|x\|^2, \ \forall x, y \in H. \tag{1}$$

Then we can construct the following variational inequality problem with viscosity. Find $x^* \in C$ such that

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \ \forall x \in F(T), \tag{2}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \left\{\frac{1}{2}\langle Ax, x \rangle - h(x)\right\},$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$), and γ is a suitable positive constant.

Recall that a mapping $T : K \to K$ is said to be a strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{3}$$

for all $x, y \in K$ (if (3) holds, we also say that T is a k-strict pseudo-contraction).

The concept of strong convergence of iterative schemes for family of mapping and study on variational inequality problem have been argued extensively. Recently, some results with a special flaw in the step of proof to reach (2) have been observed which needs to be reconsidered and corrected. The existence of this error which needs a meticulous look to be seen motivates us to fix it and also warn the researchers to take another path when arriving at the mentioned step of proof.

2 Some Iterative Processes for a Finite Family of Strict Pseudo-contractions

In this section, focusing on the strong convergence theorems of iterative process for a finite family of strict pseudo-contractions, we list the main results of some recent articles which all utilized a same procedure (with a flaw) in a part of the proof. In order to amend the observed flaw we ignore some paragraphs in the corresponding proofs and fill them by the computations extracted by our simple technique.

In 2009, Qin et al. [3] presented the following nice result. They obtained a strong convergence theorem of modified Mann iterative process for strict pseudo-contractions in Hilbert space H. The sequence $\{x_n\}$ was defined by

$$\begin{cases} x_1 = x \in K, \\ y_n = P_k[\beta_n x_n + (1-\beta_n)Tx_n], \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \ \forall n \geq 1. \end{cases} \tag{4}$$

Theorem 1 ([3]). *Let K be a closed convex subset of a Hilbert space H such that $K + K \subset K$ and $f \in \prod_K$ with the coefficient $0 < \alpha < 1$. Let A be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and let $T : K \to H$ be a k-strictly pseudo-contractive non-selfmapping such that $F(T) \neq \emptyset$. Given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$, the following control conditions are satisfied*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $k \leq \beta_n \leq \lambda < 1$ *for all* $n \geq 1$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by the composite process (4) Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F(T)$, which also solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

In the proof of Theorem 1, in order to prove

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0, \quad \text{(see (2.15) in [3])}, \tag{5}$$

where x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)P_K S x_t$, using (1) the authors obtained the following inequality

$$((\bar{\gamma}t)^2 - 2\bar{\gamma}t)\|x_t - x_n\|^2 \leq (\bar{\gamma}t^2 - 2t)\langle A(x_t - x_n), x_t - x_n \rangle$$

which is obviously impossible for $0 < t < \frac{2}{\bar{\gamma}}$. We remark that t is supposed to be vanished in the next step of proof. Here, by ignoring the computations (2.10)–(2.14) in [3] we suggest a new way to show (5) without any new condition. First let us recall the following concepts.

Definition 1. Let (X,d) be a metric space and K be a nonempty subset of X. For every $x \in K$, the distance between the point x and K is denoted by $d(x,K)$ and is defined by the following minimization problem:

$$d(x,K) := \inf d(x,y).$$

The *metric projection operator*, also said to be the nearest point mapping onto the set K is the mapping $P_K : X \to 2^K$ defined by

$$P_K(x) := \{z \in K : d(x,z) = d(x,K)\}, \quad \forall x \in X.$$

If $P_K(x)$ is singleton for every $x \in X$, then K is said to be a *Chebyshev set.*

Definition 2 ([4])**.** We say that a metric space (X,d) has property (P) if the metric projection onto any Chebyshev set is a nonexpansive mapping.

For example, any CAT(0) space has property (P). Bring in mind that Hadamard space (i.e., complete CAT(0) space) is a non-linear generalization of a Hilbert space. In the literature they are also equivalently defined as complete CAT(0) spaces.

Now, we are in a position to prove (5).

Proof. To prove inequality (5) we first find an upper bound for $\|x_t - x_n\|^2$ as follows.

$$\begin{aligned}\|x_t - x_n\|^2 &= \langle x_t - x_n, x_t - x_n\rangle \\ &= \langle t\gamma f(x_t) + (I - tA)P_K Sx_t - x_n, x_t - x_n\rangle \\ &= \langle t(\gamma f(x_t) - Ax_t) + t(Ax_t - AP_K Sx_t) \\ &\quad + (P_K Sx_t - P_K Sx_n) + (P_K Sx_n - x_n), x_t - x_n\rangle \\ &\le t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + t\|A\| \cdot \|x_t - P_K Sx_t\| \cdot \|x_t - x_n\| \\ &\quad + \|x_t - x_n\|^2 + \|P_K Sx_n - x_n\| \cdot \|x_t - x_n\|.\end{aligned} \tag{6}$$

We remark that following argument in the proof [3, Theorem 2.1] S is nonexpansive, on the other hand, since H has property (P) hence P_K is nonexpansive and $P_K S$ is so. Now, (6) implies that

$$\begin{aligned}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle &\le \|A\| \cdot \|x_t - P_K Sx_t\| \cdot \|x_t - x_n\| \\ &\quad + \frac{1}{t}\|P_K Sx_n - x_n\| \cdot \|x_t - x_n\| \\ &= t\|A\| \cdot \|\gamma f(x_t) - AP_K Sx_t\| \cdot \|x_t - x_n\| \\ &\quad + \frac{1}{t}\|P_K Sx_n - x_n\| \cdot \|x_t - x_n\| \\ &\le tM\|A\| \cdot \|\gamma f(x_t) - AP_K Sx_t\| + \frac{M}{t}\|P_K Sx_n - x_n\|\end{aligned} \tag{7}$$

where $M > 0$ is an appropriate constant such that $M \ge \|x_t - x_n\|$ for all $t \in (0, \|A\|^{-1})$ and $n \ge 1$ (we underline that according to [5, Proposition 3.1], the map $t \mapsto x_t$, $t \in (0, \|A\|^{-1})$ is bounded).

Therefore, firstly, utilizing (2.8) in [3], taking upper limit as $n \to \infty$, and then as $t \to 0$ in (7), we obtain that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \le 0. \tag{8}$$

and the claim is proved.

In what follows we concentrate on a novel result of Marino et al. [6]. They derived a strong convergence theorem of the modified Mann iterative method for strict pseudo-contractions in Hilbert space H as follows.

Theorem 2 ([6]). *Let H be a Hilbert space and let T be a k-strict pseudo-contraction on H such that $F(T) \neq \emptyset$ and f be an α-contraction. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$, satisfying the following conditions*

(i) $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;
(iii) $0 \le k \le \beta_n \le \beta < 1$ *for all* $n \ge 1$;

let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined by the composite process

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n, \ \forall n \ge 1. \end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ strongly converge to the fixed point q of T which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q\rangle \le 0, \quad \forall p \in F(T).$$

Similar to the arguments for Theorem 1, by ignoring the parts (2.10)–(2.14) in the proof of Theorem 2 we easily obtain the following conclusion.

Proof. Since x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Bx_t$ we get

$$\begin{aligned} \|x_t - x_n\|^2 &= \langle x_t - x_n, x_t - x_n\rangle \\ &= \langle t\gamma f(x_t) + (I - tA)Bx_t - x_n, x_t - x_n\rangle \\ &= \langle t(\gamma f(x_t) - Ax_t) + t(Ax_t - ABx_t) \\ &\quad + (Bx_t - Bx_n) + (Bx_n - x_n), x_t - x_n\rangle \\ &\le t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + t\|A\|\cdot\|x_t - Bx_t\|\cdot\|x_t - x_n\| \\ &\quad + \|x_t - x_n\|^2 + \|Bx_n - x_n\|\cdot\|x_t - x_n\| \end{aligned} \tag{9}$$

where here we used the fact that $B = kI + (1-k)T$ is a nonexpansive mapping (see [7, Theorem 2]). Now, (9) implies that

$$\begin{aligned}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle &\le \|A\| \cdot \|x_t - Bx_t\| \cdot \|x_t - x_n\| \\ &\quad + \frac{1}{t}\|Bx_n - x_n\| \cdot \|x_t - x_n\| \\ &= t\|A\| \cdot \|\gamma f(x_t) - ABx_t\| \cdot \|x_t - x_n\| \\ &\quad + \frac{1}{t}\|Bx_n - x_n\| \cdot \|x_t - x_n\| \\ &\le tM\|A\| \cdot \|\gamma f(x_t) - ABx_t\| + \frac{M}{t}\|Bx_n - x_n\|\end{aligned} \tag{10}$$

where $M > 0$ is an appropriate constant such that $M \ge \|x_t - x_n\|$ for all $t \in (0, \|A\|^{-1})$ and $n \ge 1$. On the other hand since $\|Bx_n - x_n\| = (1-k)\|Tx_n - x_n\|$, by using (2.8) in [6] and taking upper limit as $n \to \infty$ at first, and then as $t \to 0$ in (10), we arrive at (8) and again the claim is proved.

In 2010, Cai and Hu [8] obtained a nice strong convergence theorem of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach space as follows.

Theorem 3 ([8]). *Let E be a real q-uniformly smooth, strictly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* and C is a closed convex subset E which is also a sunny nonexpansive retraction of E such that $C + C \subset C$ with the coefficient $0 < \alpha < 1$. Let A be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $T_i : C \to E$ be λ_i-strictly pseudo-contractive non-self-mapping such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda = \min\{\lambda_i : 1 \le i \le N\}$. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C\left[\beta_n x_n + (1-\beta_n)\sum_{i=1}^N \eta_i^{(n)} T_i x_n\right], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n A)y_n, \ \forall n \ge 1, \end{cases}$$

where f is a contraction, the sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{\gamma_n\}_{n=0}^\infty$ are in $[0,1]$, assume for each n, $\{\eta_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all n and $\eta_i^{(n)} > 0$ for all $1 \le i < N$. They satisfy the conditions (i)-(iv) of [8, Lemma 2.1] and add to the condition (v) $\gamma_n = O(\alpha_n)$. Then $\{x_n\}$ converges strongly to $z \in F$, which also solves the following variational inequality

$$\langle \gamma f(z) - Az, J(p-z)\rangle \le 0, \quad \forall p \in F.$$

Proof. Ignoring (2.8)–(2.12) in the proof of Theorem 3 (i.e., [8, Theorem 2.2]) and using the same technique as before we see

$$\begin{aligned}
\|x_t - x_n\|^2 &= \langle x_t - x_n, J(x_t - x_n)\rangle \\
&= \langle t\gamma f(x_t) + (I - tA)P_C S x_t - x_n, J(x_t - x_n)\rangle \\
&= \langle t(\gamma f(x_t) - Ax_t) + t(Ax_t - AP_C Sx_t) \\
&\quad + (P_C Sx_t - P_C Sx_n) + (P_C Sx_n - x_n), J(x_t - x_n)\rangle \\
&\le t\langle \gamma f(x_t) - Ax_t, J(x_t - x_n)\rangle + t\|A\| \cdot \|x_t - P_C Sx_t\| \cdot \|x_t - x_n\| \\
&\quad + \|x_t - x_n\|^2 + \|P_C Sx_n - x_n\| \cdot \|x_t - x_n\|
\end{aligned} \tag{11}$$

where x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)P_C Sx_t$. Again, we remark that $P_C S$ is nonexpansive and hence

$$\begin{aligned}
&\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \\
&\le \|A\| \cdot \|x_t - P_C Sx_t\| \cdot \|x_t - x_n\| \\
&\quad + \frac{1}{t}\|P_C Sx_n - x_n\| \cdot \|x_t - x_n\| \\
&= t\|A\| \cdot \|\gamma f(x_t) - AP_C Sx_t\| \cdot \|x_t - x_n\| \\
&\quad + \frac{1}{t}\|P_C Sx_n - x_n\| \cdot \|x_t - x_n\| \\
&\le tM\|A\| \cdot \|\gamma f(x_t) - AP_C Sx_t\| + \frac{M}{t}\|P_C Sx_n - x_n\|
\end{aligned} \tag{12}$$

where $M > 0$ is a proper constant such that $M \ge \|x_t - x_n\|$ for $t \in (0, \|A\|^{-1})$ and $n \ge 1$. Thus, taking upper limit as $n \to \infty$ at first, and then as $t \to 0$ in (12), the following yields

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \le 0. \tag{13}$$

Finally, in the last part of this section we focus on the main result of Kangtunyakarn and Suantai [9].

Theorem 4 ([9]). *Let H be a Hilbert space, let f be an α-contraction on H and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar\gamma > 0$. Assume that $0 < \gamma < \frac{\bar\gamma}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contraction of H into itself for some $\kappa_i \in [0,1)$ and $\kappa = \max\{\kappa_i : i = 1, 2, \cdots, N\}$ with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$. Let S_n be the S-mappings generated by $T_1, T_2, \cdots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \le \alpha_1^{n,j}$, $\alpha_3^{n,j} \le b < 1$ for all $j = 1, 2, \cdots, N-1$, $\kappa < c \le \alpha_1^{n,N} \le 1$, $\kappa \le \alpha_3^{n,N} \le d < 1$, $\kappa \le \alpha_2^{n,j} \le e < 1$ for all $j = 1, 2, \cdots, N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)S_n x_n, \\ x_{n+1} = \alpha_n\gamma(a_n u + (1 - a_n)f(x_n)) + (I - \alpha_n A)y_n, \ \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{a_n\}$ *are the sequences in* $[0,1]$. *Assume that the following conditions hold:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} a_n = 0$;
(ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j \in \{1,2,\cdots,N\}$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;
(iii) $0 \leq \kappa \leq \beta_n < \theta < 1$ *for all* $n \geq 1$ *and some* $\theta \in (0,1)$.

Then both $\{x_n\}$ *and* $\{y_n\}$ *strongly converge to* $q \in \bigcap_{i=1}^{N} F(T_i)$, *which solves the following variational inequality*

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{N} F(T_i).$$

Proof. In the proof of Theorem 4 (i.e., [9, Theorem 3.1]), leaving the inequlities (3.9)–(3.10) behind and applying the same technique as mentioned before we derive

$$\begin{aligned}
\|x_t - x_n\|^2 &= \langle x_t - x_n, x_t - x_n \rangle \\
&= \langle t\gamma f(x_t) + (I - tA)S_n x_t - x_n, x_t - x_n \rangle \\
&= \langle t(\gamma f(x_t) - Ax_t) + t(Ax_t - AS_n x_t) \\
&\quad + (S_n x_t - S_n x_n) + (S_n x_n - x_n), x_t - x_n \rangle \\
&\leq t\langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + t\|A\| \cdot \|x_t - S_n x_t\| \cdot \|x_t - x_n\| \\
&\quad + \|x_t - x_n\|^2 + \|S_n x_n - x_n\| \cdot \|x_t - x_n\|
\end{aligned} \tag{14}$$

where x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)S_n x_t$. Here, we notify that S_n is nonexpansive and hence

$$\begin{aligned}
&\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \\
&\leq \|A\| \cdot \|x_t - S_n x_t\| \cdot \|x_t - x_n\| + \frac{1}{t}\|S_n x_n - x_n\| \cdot \|x_t - x_n\| \\
&= t\|A\| \cdot \|\gamma f(x_t) - AS_n x_t\| \cdot \|x_t - x_n\| \\
&\quad + \frac{1}{t}\|S_n x_n - x_n\| \cdot \|x_t - x_n\| \\
&\leq tM\|A\| \cdot \|\gamma f(x_t) - AS_n x_t\| + \frac{M}{t}\|S_n x_n - x_n\|
\end{aligned} \tag{15}$$

where $M > 0$ is a proper constant such that $M \geq \|x_t - x_n\|$ for $t \in (0, \|A\|^{-1})$ and $n \geq 1$. Thus, following (3.8) in [9], taking upper limit as $n \to \infty$ at first, and then as $t \to 0$ in (15), the following yields

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0$$

and the claim is proved.

3 General Iterative Scheme for Semigroups of Uniformly Asymptotically Regular Nonexpansive Mappings

Throughout this section, we focus on the main result of Yang [10] as follows. First, we recall that a continuous operator of the semigroup $\mathscr{T} = \{T(t) : 0 \leq t < \infty\}$ is said to be uniformly asymptotically regular (u.a.r.) on K if for all $h \geq 0$ and any bounded subset C of K, $\lim_{t\to\infty} \sup_{x\in C} \|T(h)T(t)x - T(t)x\| = 0$.

Theorem 5 ([10]). *Let K be a nonempty closed convex subset of a reflexive, smooth and strictly convex Banach space E with a uniformly Gâteaux differentiable norm. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a uniformly asymptotically regular nonexpansive semigroup on K such that $F(\mathscr{T}) \neq \emptyset$, and $f \in \Pi_K$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)T(t_n)x_n,$$

such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, the given sequences $\{x_n\}$ and $\{\delta_n\}$ are in $(0,1)$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(iii) $h, t_n \geq 0$ *such that* $t_{n+1} - t_n = h$ *and* $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly to q, as $n \to \infty$, q is the element of $F(\mathscr{T})$ such that q is the unique solution in $F(\mathscr{T})$ to the variational inequality

$$\langle (A - \gamma f)q, j(q - z)\rangle \leq 0, \quad \forall z \in F(\mathscr{T}).$$

Proof. Ignoring (3.15)–(3.17) in the proof of [10, Theorem 3.5] and using the same technique as before we see that

$$\begin{aligned}
\|u_m - x_n\|^2 =& \langle u_m - x_n, j(u_m - x_n)\rangle \\
=& \langle \alpha_m \gamma f(u_m) + (I - \alpha_m A)S(t_m)u_m - x_n, j(u_m - x_n)\rangle \\
=& \langle \alpha_m(\gamma f(u_m) - Au_m) + \alpha_m(Au_m - AS(t_m)u_m) \\
&+ (S(t_m)u_m - S(t_m)x_n) + (S(t_m)x_n - x_n), j(u_m - x_n)\rangle \\
\leq& \alpha_m \langle \gamma f(u_m) - Au_m, j(u_m - x_n)\rangle + \alpha_m \|A\| \\
&\cdot \|u_m - S(t_m)u_m\| \cdot \|u_m - x_n\| + \|u_m - x_n\|^2 \\
&+ \|S(t_m)x_n - x_n\| \cdot \|u_m - x_n\|
\end{aligned} \tag{16}$$

where $u_m \in K$ is the unique solution of the fixed point problem $u_m = \alpha_m \gamma f(u_m) + (I - \alpha_m A)S(t_m)u_m$. It is worth mentioning that $\mathscr{S} := \{S(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mapping and this helped us to find the upper bound of (16). Furthermore,

$$\begin{aligned}\langle Au_m - \gamma f(u_m), j(u_m - x_n)\rangle &\le \|A\| \cdot \|u_m - S(t_m)u_m\| \cdot \|u_m - x_n\| \\ &\quad + \frac{1}{\alpha_m}\|S(t_m)x_n - x_n\| \cdot \|u_m - x_n\| \\ &= \alpha_m\|A\| \cdot \|\gamma f(u_m) - AS(t_m)u_m\| \cdot \|u_m - x_n\| \\ &\quad + \frac{1}{\alpha_m}\|S(t_m)x_n - x_n\| \cdot \|u_m - x_n\| \\ &\le \alpha_m M\|A\| \cdot \|\gamma f(u_m) - AS(t_m)u_m\| \\ &\quad + \frac{M}{\alpha_m}\|S(t_m)x_n - x_n\|\end{aligned} \tag{17}$$

where $M > 0$ is a proper constant such that $M \ge \|u_m - x_n\|$ for $m, n \in \mathbb{N}$. Thus, following (i), (3.14) in [10], taking upper limit as $n \to \infty$ at first, and then as $m \to \infty$ in (17), the following yields

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle Au_m - \gamma f(u_m), j(u_m - x_n)\rangle \le 0 \tag{18}$$

which again proves our claim.

Remark 1. In view of the technique of the proof as above and the ones in the former section, one can easily see that we did not utilize (1) as an important property of the strongly positive bounded linear operator A. It is worth pointing out this property is crucial for the aforementioned results and we reduced the dependence of results to the property (1); we refer reader to see, for instance, (2.12) in [3], (2.10) in [8], (2.12) in [6], (3.16) in [10] and the inequalities right after (3.9) in [9].

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